Starter

$$y = x^2 + 3x^{\frac{1}{2}}.$$

a Find $\dfrac{dy}{dx}$.

b Show that $2x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6x = 0$.

Starter - answers

a $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \frac{3}{2}x^{-\frac{1}{2}}$

b $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - \frac{3}{4}x^{-\frac{3}{2}}$

$$\begin{aligned}
\therefore\ 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6x
&= 2x(2 - \tfrac{3}{4}x^{-\frac{3}{2}}) + 2x + \tfrac{3}{2}x^{-\frac{1}{2}} - 6x \\
&= 4x - \tfrac{3}{2}x^{-\frac{1}{2}} + 2x + \tfrac{3}{2}x^{-\frac{1}{2}} - 6x \\
&= 0
\end{aligned}$$

F Exponentials and logarithms

F6 Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- reduce a non-linear relationship to linear form
- plot a graph from given data, drawing a line of best fit by eye and using it to calculate the gradient and intercept to estimate for unknown constants.

Note: this is an essential skill in A-level sciences and there is an ideal opportunity here to link to real data: power laws for relationships of the form $y = ax^n$ and exponential laws for those of the form $y = kb^x$

https://sites.google.com/view/tlmaths/home/a-level-maths/as-only/f-exponentials-logarithms/f6-reduction-to-linear-form

5.4 Curve Fitting

Two common non-linear relationships are:

- Polynomial growth: where and are constants
- Exponential growth: where and are constants

5.4 Curve Fitting

Plotting these relationships where the scales are linear means the graphs will become very steep very quickly and will become difficult to read with any accuracy.

Linear graphs are much easier to deal with and we can easily read off values, find the gradient of the line etc…

Using logs we can convert a non-linear relationship into a linear one.

5.4 Curve Fitting

Polynomial Relationships

$$y = a x^n$$

$$\log y = \log\left(a x^n\right)$$

$$\log y = \log a + \log x^n$$

$$\log y = n \log x + \log a$$

5.4 Curve Fitting

$$\log y = n \log x + \log a$$

This can be written in linear form and plotted to give a straight line where

$$\boxed{Y = mX + C}$$

$$Y = \log y$$

$$X = \log x$$

$$Gradient\ m = n$$

$$Intercept\ C = \log a$$

5.4 Curve Fitting

Example 1

The number of employees, , working for a company t years after it was founded can be modelled by the equation . The table below shows the number of employees the company has:

Age of company, t (years)	2	5	8	13	25
No. of employees, p	3	7	10	16	29

Plot a linear graph to represent this data, and use this to find the values of and .

5.4 Curve Fitting

Example 1

Add two rows to the table of values:

log t and log p

Plot the graph and draw a line of best f

$$p = a t^b$$

$$\log p = \log\left(a t^b\right)$$

$$\log p = \log a + \log t^b$$

$$\log p = b \log t + \log a$$

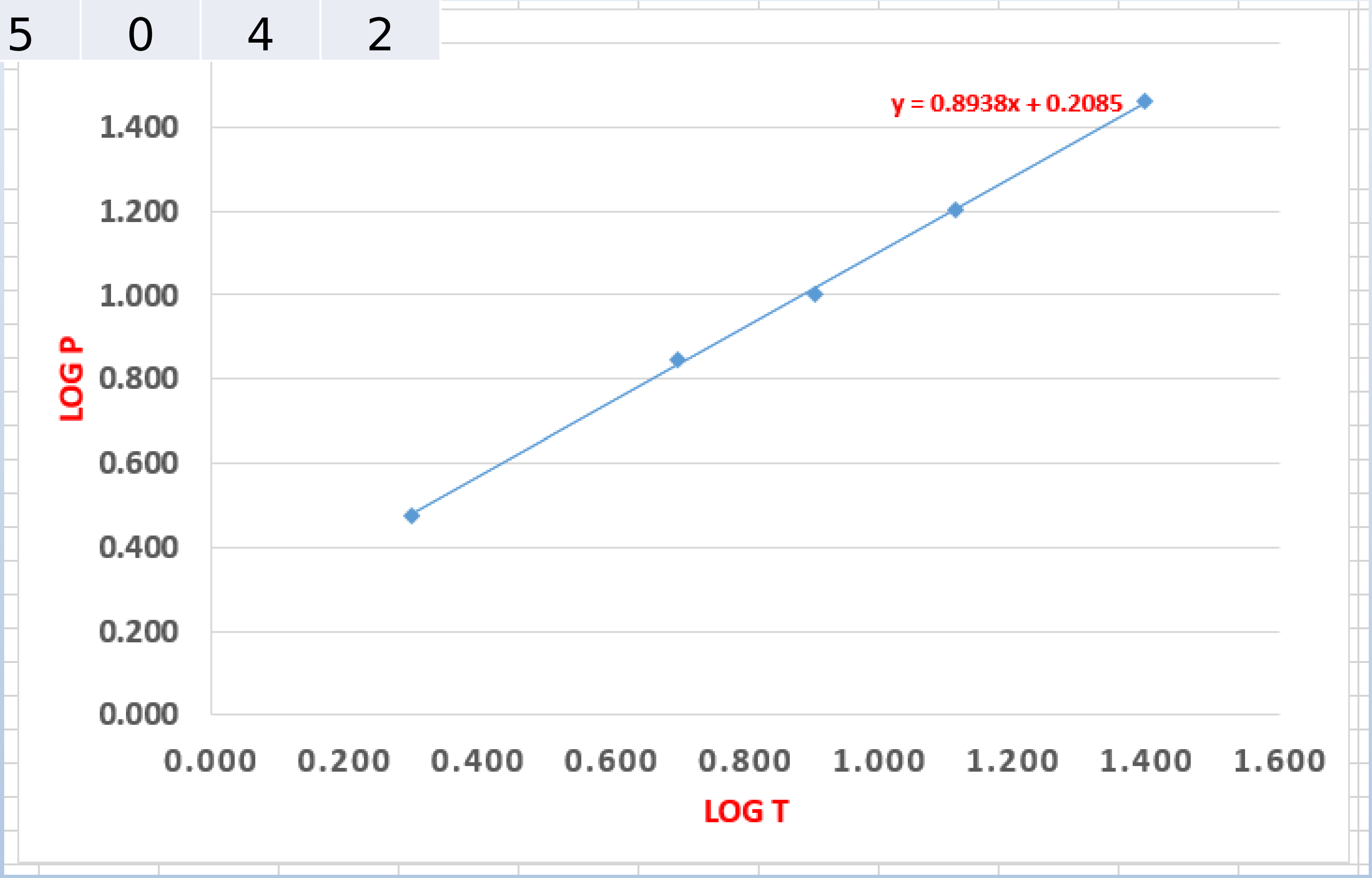
Y¿$mX+$¿C¿

t	2	5	8	13	25
p	3	7	10	16	29
log t	0.301	0.699	0.903	1.114	1.398
log p	0.477	0.845	1	1.204	1.462

5.4 Curve Fitting

log t	0.301	0.699	0.903	1.114	1.398
log p	0.477	0.845	1.000	1.204	1.462

$$\log p = b \log t + \log a$$

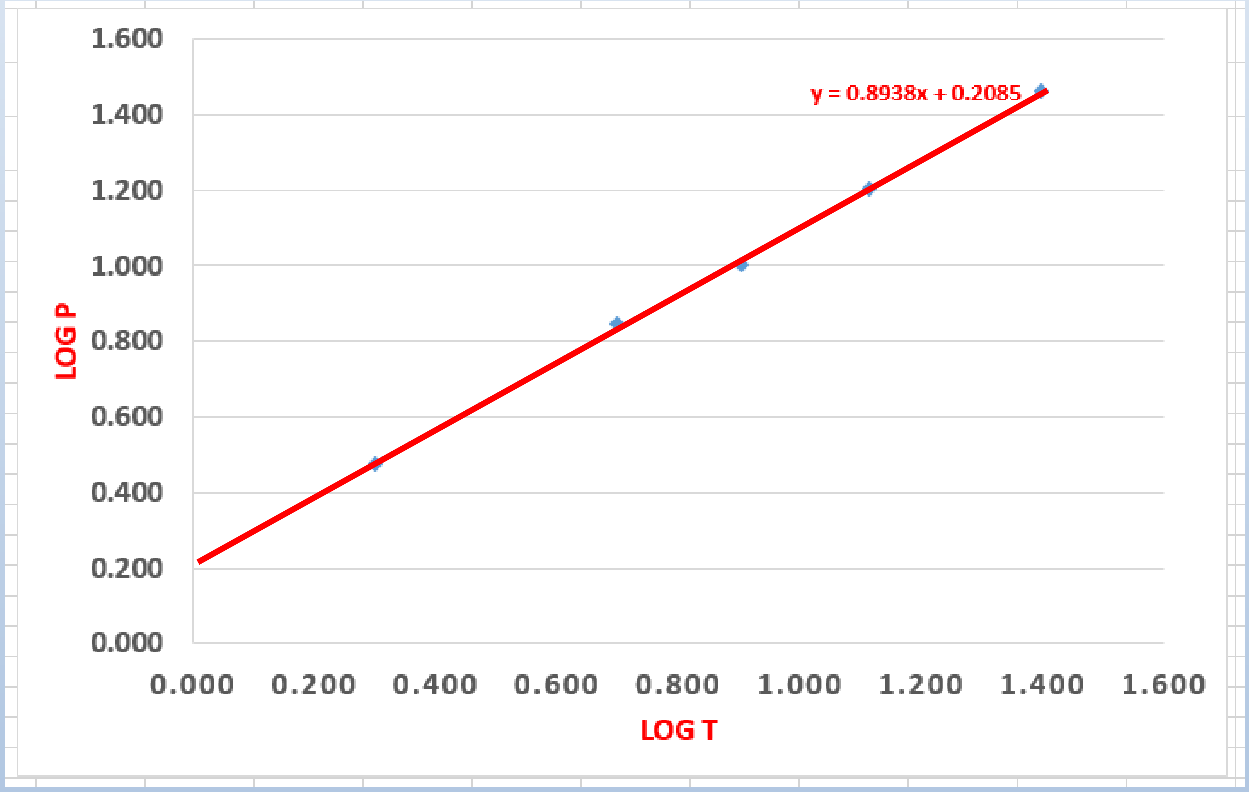


5.4 Curve Fitting

From the graph: $\log p = b \log t + \log a$

Vertical intercept:

(3dp)

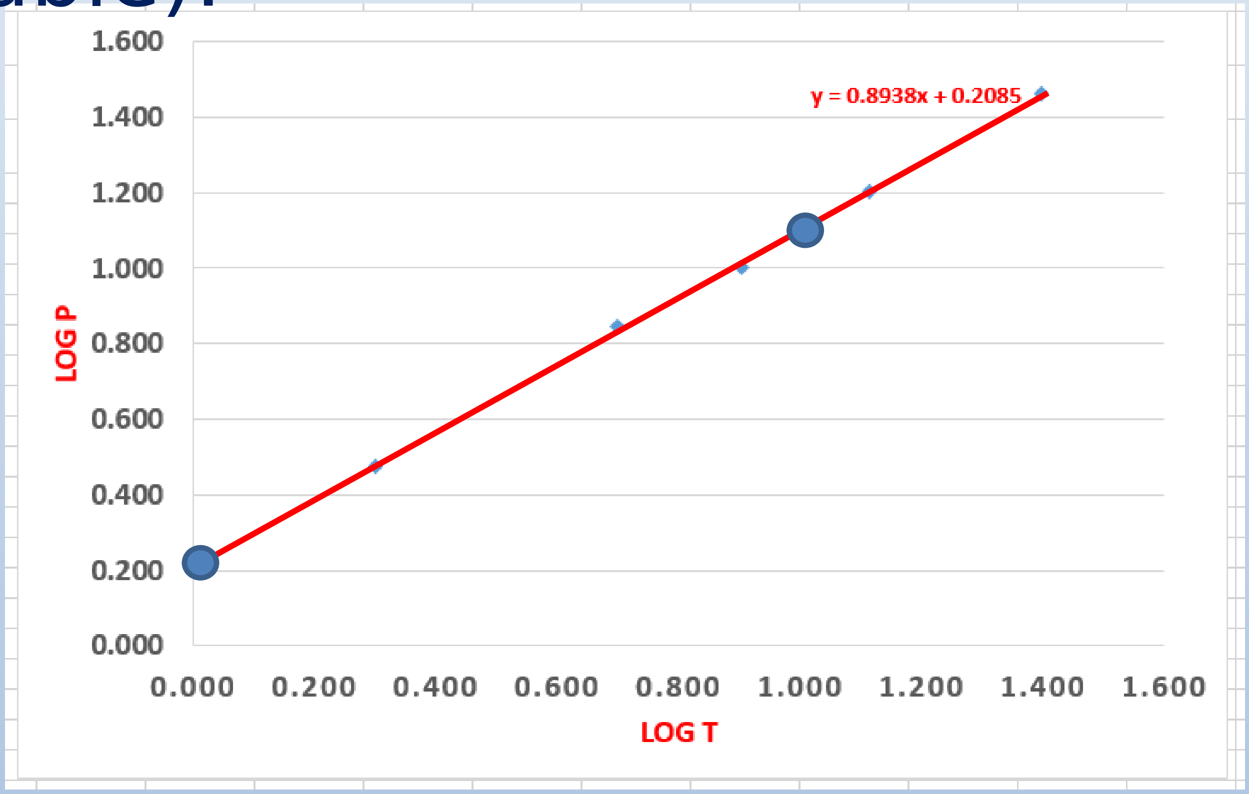


5.4 Curve Fitting

From the graph: $\log p = b \log t + \log a$

Choose two points on the line of best fit (or from the table):

$$\left(1.0, 1.1\right)$$



5.4 Curve Fitting

$$p = a t^{b}$$

From the graph: $a = 1.585 \qquad b = 0.9$

$$\therefore p = 1.585\, t^{0.9}$$

Age of company, t (years)	2	5	8	13	25
No. of employees, p	3	7	10	16	29

Check:
(to 1sf)

5.4 Curve Fitting

Example 2a

A scientist grows a certain type of bacteria in a laboratory. The table shows the number of bacteria measured at different times of the day.

Time	8:00am	10:00am	1:00pm	3:00pm	8:00pm	10:00pm
Number of bacteria, P	20	86	283	481	1219	1511

This growth may be modelled by an equation of the form $P = at^b$, where P is the population of bacteria, t is the number of hours after 6:00am, and a and b are constants to be determined.

a. Show that, according to this model, the graph of $\log_{10} P$ against $\log_{10} t$ should be a straight line of gradient b. State, in terms of a, the intercept on the vertical axis.

$$P = at^b \Rightarrow \log_{10} P = \log_{10}(at^b)$$
$$\Rightarrow \log_{10} P = \log_{10} a + \log_{10} t^b$$
$$\Rightarrow \log_{10} P = \log_{10} a + b \log_{10} t$$

5.4 Curve Fitting

Example 2b

A scientist grows a certain type of bacteria in a laboratory. The table shows the number of bacteria measured at different times of the day.

Time	8:00am	10:00am	1:00pm	3:00pm	8:00pm	10:00pm
Number of bacteria, P	20	86	283	481	1219	1511

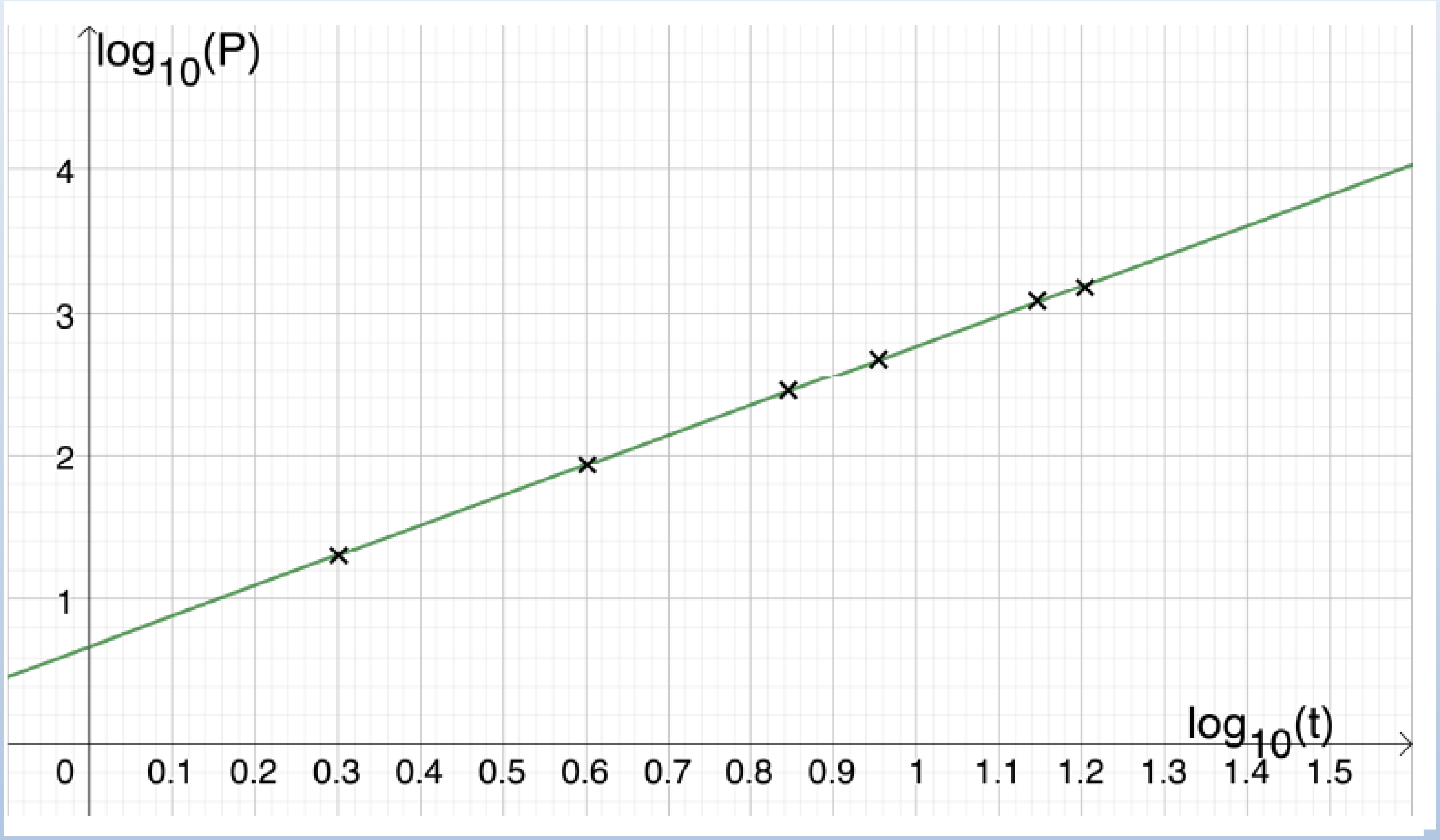
This growth may be modelled by an equation of the form $P = at^b$, where P is the population of bacteria, t is the number of hours after 6:00am, and a and b are constants to be determined.

b. Complete the table of values below, writing your answers to 2 decimal places, and plot $\log_{10} P$ against $\log_{10} t$. Draw by eye a line of best fit for the data.

t	2	4	7	9	14	16
P	20	86	283	481	1219	1511
$\log_{10} t$	0.30103	0.60206	0.8451	0.95424	1.14613	1.20412
$\log_{10} P$	1.30103	1.9345	2.45179	2.68215	3.086	3.17926

5.4 Curve Fitting

Example 2b



5.4 Curve Fitting

Example 2c

A scientist grows a certain type of bacteria in a laboratory. The table shows the number of bacteria measured at different times of the day.

Time	8:00am	10:00am	1:00pm	3:00pm	8:00pm	10:00pm
Number of bacteria, P	20	86	283	481	1219	1511

This growth may be modelled by an equation of the form $P = at^b$, where P is the population of bacteria, t is the number of hours after 6:00am, and a and b are constants to be determined.

c. Use your graph to find the equation for P in terms of t.

Gradient ≈ 2.095

Intercept ≈ 0.675

$$P = 4.727 \times t^{2.095}$$

5.4 Curve Fitting

Example 2d

A scientist grows a certain type of bacteria in a laboratory. The table shows the number of bacteria measured at different times of the day.

Time	8:00am	10:00am	1:00pm	3:00pm	8:00pm	10:00pm
Number of bacteria, P	20	86	283	481	1219	1511

This growth may be modelled by an equation of the form $P = at^b$, where P is the population of bacteria, t is the number of hours after 6:00am, and a and b are constants to be determined.

d. Estimate the number of bacteria at 5:00pm.

$$P|_{t=11} \approx 718$$

Complete the remaining three questions.